NONSTATIONARY HEAT CONDUCTION IN A HALF-SPACE WITH AN INFINITE NUMBER OF CYLINDRICAL HEAT SOURCES

I. A. Ioffe

UDC 536.2.01

The problem of nonstationary heat conduction is treated for a half-space containing an infinite number of cylindrical heat sources, and a boundary condition of the first kind at the surfaces. It is assumed that the radii of the sources are small in comparison with their spacing and the ordinate of the center.

1. The problem under consideration is a mathemetical model of nonstationary heat transfer between a semibounded massive body and a number of cooling or heating pipes embedded in it. Practical examples of this model are such widely used heat-transfer systems as ground heating of greenhouses and open ground, certain radiant heating systems, heating floors of buildings, etc.

The corresponding steady-state problem was partially solved in [1] (the formula for the rate of flow of heat). The electrical simulation of the steady-state temperature distribution is described in [2], and an analytic expression for it was obtained subsequently in [3, 4].

In the mathematical formulation the nonstationary problem is reduced to the solution of the heat-conduction equation for a uniform half-space $y \geq 0$ (Fig. 1) containing an infinite number of cylindrical heat sources with parameters ρ, y_0, S for the following boundary conditions:

$$t(x_*, y_*, 0) = t_0 \tag{1.1}$$
$$t(x_*, 0, \tau) = t_0 \tag{1.2}$$
$$t(x_*, \infty, \tau) = t_0 \tag{1.3}$$
$$t(x_*, y_*, \tau)\,|_{x_*, y_* \in \Gamma} = t_T \tag{1.4}$$

where t is the temperature at the point x_*, y_* at time τ, t_0 is the initial temperature, and t_T is the temperature at the surface Γ of the sources.

The familiar relation [5]

$$t(r_*, \tau) = \frac{\varphi_*}{4\pi a\tau} \exp\left[-\frac{\rho_*^2 + r_*^2}{4a\tau}\right] I_0\left(\frac{\rho_* r_*}{2a\tau}\right)$$

describes the spatial temperature distribution produced by an instantaneous cylindrical surface heat source of strength φ_* and radius ρ_*. Here a is the thermal diffusivity of the body, and r_* is the radius vector to a point.

We assume that the source strength varies with time. Then the integral

$$T(r_*, \tau) = \frac{1}{4\pi a\tau}\int_0^\tau \varphi_*(u) \exp\left(-\frac{\rho_*^2 + r_*^2}{4a(\tau - u)}\right) I_0\left(\frac{\rho_* r_*}{2a(\tau - u)}\right) \frac{du}{\tau - u} \tag{1.5}$$

will be the temperature function of a continuously active cylindrical heat source of variable strength.

We consider the following expression in dimensionless variables for the results of superposing an infinite number of sources and corresponding sinks of the type (1.5)

Leningrad. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 164-168, July-August, 1974. Original article submitted August 20, 1973.

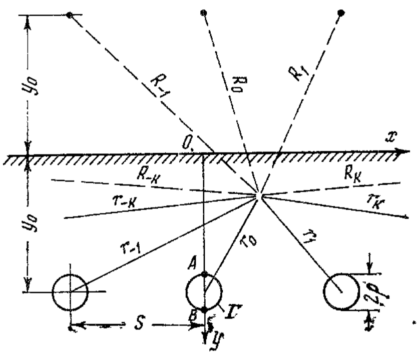

Fig. 1

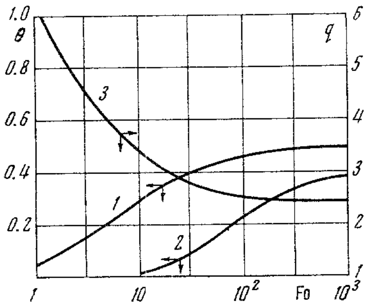

Fig. 2

$$\theta(x, y, \mathrm{Fo}) = \frac{1}{4\pi}\sum_{k=-\infty}^{\infty}\int_0^{\mathrm{Fo}} \varphi(u)\left[\exp\left(-\frac{\rho^2 + r_k^2}{4(\mathrm{Fo}-u)}\right) I_0\left(\frac{\rho r_k}{2(\mathrm{Fo}-u)}\right) - \exp\left(-\frac{\rho^2 + R_k^2}{4(\mathrm{Fo}-u)}\right) I_0\left(\frac{\rho R_k}{2(\mathrm{Fo}-u)}\right)\right]\frac{du}{\mathrm{Fo}-u} \tag{1.6}$$

$$\theta = \frac{t(x, y, \tau) - t_0}{t_T - t_0}, \qquad \mathrm{Fo} = \frac{a\tau}{S^2}$$
$$r_k = [(k-x)^2 + (y-y_0)^2]^{1/2}, \qquad R_k = [(k-x)^2 + (y+y_0)^2]^{1/2} \tag{1.7}$$

Here ρ, y_0, x, y are respectively the dimensionless radius of the sources, the ordinate of their centers, and the linear coordinates for a given dimension S. In (1.6) and (1.7) the subscript $k = 0, \pm 1, \ldots, \pm\infty$ takes on positive values for sources and sinks in the region $x > 0$, and negative values in the region $x < 0$.

Equation (1.6), as a linear sum of integrals of the heat-conduction equation, satisfies that equation and, taking account of the transformation to dimensionless variables, the boundary conditions (1.1)-(1.3). The necessary and sufficient condition for the solution to be unique is that boundary condition (1.4) be satisfied, and this can be achieved by an appropriate choice of the form of the function $\varphi(\mathrm{Fo})$. Referring (1.6) to points $A(0, y_0 - \rho)$ or $B(0, y_0 + \rho)$ we find that $\varphi(\mathrm{F0})$ is determined by the following Volterra integral equation of the first kind of the convolution type:

$$1 = \frac{1}{4\pi}\sum_{k=-\infty}^{\infty}\int_0^{\mathrm{Fo}} \varphi(u)\left[\exp\left(-\frac{\rho^2 + n_k^2}{4(\mathrm{Fo}-u)}\right) I_0\left(\frac{\rho n_k}{2(\mathrm{Fo}-u)}\right) - \exp\left(-\frac{\rho^2 + N_k^2}{4(\mathrm{Fo}-u)}\right) I_0\left(\frac{\rho N_k}{2(\mathrm{Fo}-u)}\right)\right]\frac{du}{\mathrm{Fo}-u} \tag{1.8}$$

where n_k and N_k are respectively the radius vectors to the points A and B. Making the simplifying assumptions

$$2y_0 \gg \rho, \qquad \rho \ll 1 \tag{1.9}$$

and using (1.7) these are equal to

$$N_k = \sqrt{k^2 + 4y_0^2}, \quad n_k = \begin{cases} k, & |k| \geqslant 1 \\ \rho, & k = 0 \end{cases} \tag{1.10}$$

Using the convolution theorem and taking the Laplace transforms of (1.6) and (1.8) we obtain the two equations

$$\bar{\theta}(s) = \frac{\bar{\varphi}(s)}{2\pi} I_0(\rho\sqrt{s}) \sum_{k=-\infty}^{\infty} [K_0(r_k\sqrt{s}) - K_0(R_k\sqrt{s})]$$

$$\frac{1}{s} = \frac{\bar{\varphi}(s)}{2\pi} I_0(\rho\sqrt{s}) \sum_{k=-\infty}^{\infty} [K_0(n_k\sqrt{s}) - K_0(N_k\sqrt{s})]$$

where $\theta(s)$ and $\varphi(s)$ are the transforms of the respective functions. Solving these we find

$$\bar{\theta}(s) = \sum_{k=-\infty}^{\infty} [K_0(r_k\sqrt{s}) - K_0(R_k\sqrt{s})]\left\{s\sum_{k=-\infty}^{\infty} [K_0(n_k\sqrt{s}) - K_0(N_k\sqrt{s})]\right\}^{-1} \tag{1.11}$$

The inverse transform of (1.11) is found by using the formula for the Mellin inversion. In this case the faimilar Hankel contour is used with a branch cut along the negative real axis between the branch points $s = 0$ and $s = \infty$ of the integrand. By applying standard methods of contour integration and using the properties of Bessel functions we obtain

$$\theta(x, y, \mathrm{Fo}) = \theta_{st} - \frac{2}{\pi}\int_0^{\infty} e^{-z^2\mathrm{Fo}}\, \frac{J(n_k z, N_k z)\, Y(r_k z, R_k z) - J(r_k z, R_k z)\, Y(n_k z, N_k z)}{[J(n_k z, N_k z)]^2 + [Y(n_k z, N_k z)]^2}\,\frac{dz}{z} \tag{1.12}$$

$$J(b_k z, B_k z) = \sum_{k=-\infty}^{\infty} [J_0(b_k z) - J_0(B_k z)] \qquad Y(b_k z, B_k z) = \sum_{k=-\infty}^{\infty} [Y_0(b_k z) - Y_0(B_k z)] \tag{1.13}$$

$$\theta_{st} = \sum_{k=-\infty}^{\infty} \ln\frac{R_k}{r_k} \Big/ \sum_{k=-\infty}^{\infty} \ln\frac{N_k}{n_k} \tag{1.14}$$

Equation (1.14) describes the steady-state temperature distribution for the problem (3), and the series appearing in its numerator and denominator are summed in [3, 1] respectively. This enables us to write (1.14) in the form

$$\theta_{st} = \frac{1}{2} \ln \frac{\operatorname{ch} 2\pi (y_0 + y) - \cos 2\pi x}{\operatorname{ch} 2\pi (y_0 - y) - \cos 2\pi x} \quad \ln \left(\frac{1}{\pi\rho} \operatorname{sh} 2\pi y_0 \right) \tag{1.15}$$

2. Under assumptions (1.9) the temperature fields produced by sources and sinks in the neighborhood of a particular source can be assumed constant. Then the total field in the same neighborhood is cylindrical, and the rate of heat flow per unit length of the source (k = 0 for definiteness) is

$$Q = \left(-2\pi\rho\lambda \frac{\partial t}{\partial r_0} \right)_{r_0=\rho} = -2\pi\lambda\rho (t_T - t_0) \frac{\partial \theta}{\partial z_0} \Big|_{r_0=\rho} \tag{2.1}$$

In satisfying (2.1) the operations of differentiation must be kept in mind, and since the field is cylindrical

$$\frac{\partial R_k}{\partial r_0} \Big|_{r_0=\rho} = 0, \quad \frac{\partial r_k}{\partial r_0} \Big|_{r_0=\rho} = \begin{cases} 0, & |k| \geqslant 1 \\ 1, & k = 0 \end{cases} \tag{2.2}$$

Differentiating Eq. (1.12) with its stationary component in the form (1.14) and using (2.2) we find that the dimensionless rate of heat flow $q = Q/[\lambda(t_T - t_0)]$ is

$$q = 2\pi \left[\ln \left(\frac{1}{\pi\rho} \operatorname{sh} 2\pi y_0 \right) \right]^{-1} + 4\rho \int_0^\infty \exp(-z^2 \mathrm{Fo}) \quad \frac{J_1(\rho z) Y(n_k z, N_k z) - Y_1(\rho z) J(n_k z, N_k z)}{[J(n_k z, N_k z)]^2 + [Y(n_k z, N_k z)]^2} dz \tag{2.3}$$

The first term on the right-hand side of (2.3) is the dimensionless form of the familiar O. E. Vlasov formula.

Figure 2 shows the dependence of the dimensionless temperatures (curves 1 and 2) and the rate of heat flow (curve 3) on the Fourier number, calculated by Eqs. (1.12) and (2.3) for the following source parameters: $\rho = 0.04$ and $y_0 = 0.2$. In this case curves 1 and 2 illustrate the time behavior of the temperature at two points of the half-space having the dimensionless coordinates $x_1 = 0.16$, $y_1 = 0.2$ and $x_2 = 0.5$, $y_2 = 0.4$.

3. We estimate the error of the solution due to the approximate character of conditions (1.9) by using the familiar relation between a function and its transform

$$\theta(x, y, 0) = \lim_{s \to \infty} s\bar{\theta}(x, y, s)$$

Then by using the asymptotic formula

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$$

we find from (1.11)

$$\theta(x, y, 0) = \begin{cases} 1, & x, y \in \Gamma \\ 0, & x, y \notin \Gamma \end{cases} \tag{3.1}$$

which represents conditions (1.1) and (1.4) in dimensionless variables.

Thus, Eqs. (1.12) and its partial derivative (2.3) are exact solutions at zero time since for Fo = 0 the boundary conditions of the problem are satisfied rigorously. The deviation from condition (3.1) on the contour of the source under consideration begins when the temperature functions of the remaining sources and sinks take on appreciably different values in its vicinity.

We note that since function (1.5) for a source of sink increases monotonically with time, its value for any fixed point is maximum for $F_0 = \infty$. Thus it follows that the error in the solution is maximum in the steady state.

A numerical analysis [6] shows that when the conditions

$$\rho \leqslant 0.13, \qquad y_0 / \rho \geqslant 4 \tag{3.2}$$

are satisfied simultaneously the relative error of the O. E. Vlasov formula is less than 4%. The error of the steady-state component of the relative temperature for $\rho \leq 0.1$ is negligibly small [7]. Thus when the limitations (3.2) are satisfied, which is generally the case in practice, the maximum error in solutions (1.12) and (2.3) is no more than a few percent.

LITERATURE CITED

1. O. E. Vlasov, Additions of Editor G. Greber, Introduction to the Theory of Heat Transfer [in Russian], Gosenergdizdat, Moscow (1933).

2. S. S. Kutateladze and A. L. Rabinovich, "Calculation of ground heating of Greenhouse," Otopl. Vent. No. 12 (1935).
3. I. A. Ioffe," On the stationary temperature distribution in a large seimbounded mass with integral cylindrical heat sources," Zh. Tekh. Fiz., 28, No. 5 (1958).
4. A. A. Sander, "Temperature distribution of a number of ducts in a large mass," Izv. VUZ. Stroitel. Arkh. No. 1 (1958).
5. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, Oxford, New York (1964).
6. Ya. N. Kanya, "On the accuracy of replacing a cylindrical heat source by a point source for finding the temperature distribution in a large semibounded mass," Inzh.-Fiz. Zh., 82, no. 2 (1972).
7. D. A. Pereverzev, "On a two-dimensional problem of steady-state heat conduction," Inzh.-Fiz. Zh., 8, No. 5 (1965).